Convex Approximations for a Bi-level Formulation of Data-Enabled Predictive Control

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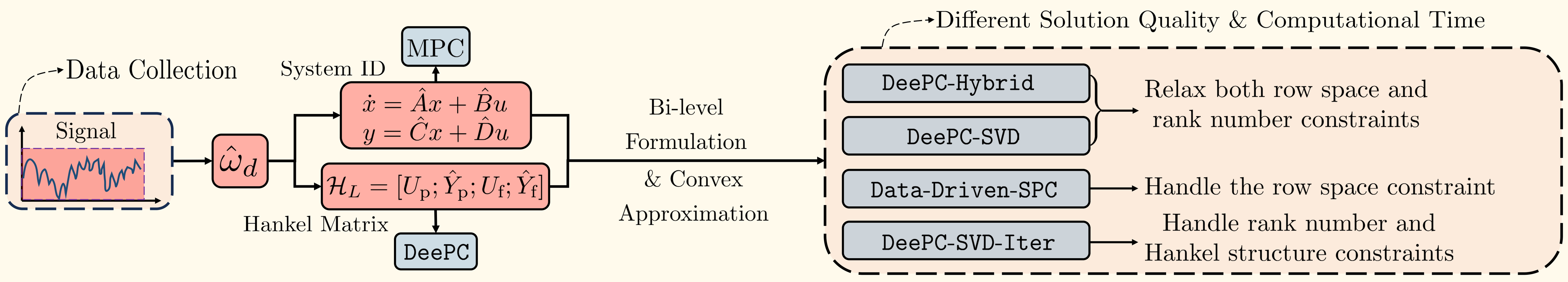
Summary



Relationships: under mild conditions, we have

- ***DeePC-Hybrid*** and ***DeePC-SVD*** have the same optimal solution.
- If λ_2 is sufficiently large, ***DeePC-Hybrid*** and ***DeePC-SVD*** have the same optimal solution as ***Data-Driven-SPC***

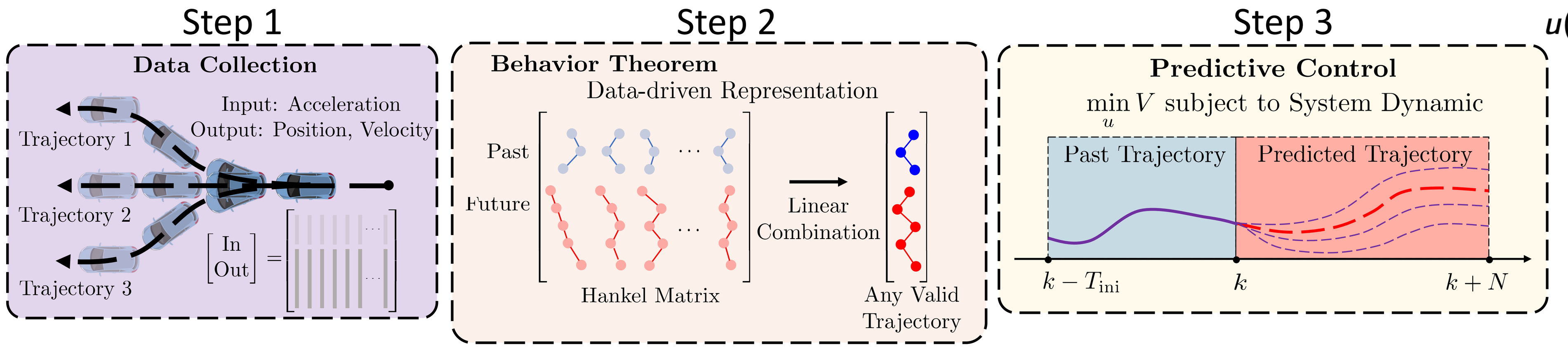
Overview

Two main data-driven control methods

- **Indirect Data-driven Control:**
  Data → model + uncertainty → control
- **Direct Data-driven Control:**
  Bypassing models, directly design controllers from data

Data-EnablEd Predictive Control (DeePC) (Coulson, 2019)

- **Combine Willem's fundamental lemma with predictive control**



- **Model Predictive Control (Indirect)**

$$\min_{x,u,y} \sum_{k=t}^{t+N-1} \left(\|y(k) - y_r(k)\|_Q^2 + \|u(k)\|_R^2\right)$$

$$\begin{aligned}\text{subject to}\quad & x(k+1) = A x(k) + B u(k), \quad k \in [t, t+N-1] \\ & y(k) = C x(k) + D u(k), \quad k \in [t, t+N-1] \\ & x(t) = x_\mathrm{ini}, \\ & u(k) \in \mathcal{U},\ y(k) \in \mathcal{Y}, \quad k \in [t, t+N-1].\end{aligned}$$

- **Data-EnablEd Predictive Control (Direct)**

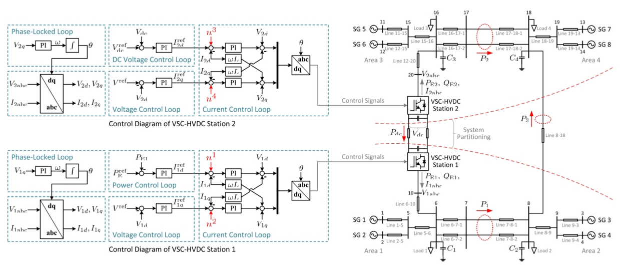
$$\min_{g,u,y} \sum_{k=t}^{t+N-1} \left(\|y(k)\|_Q^2 + \|u(k)\|_R^2\right)$$

$$\text{subject to} \quad \begin{bmatrix} U_\mathrm{P} \\ Y_\mathrm{P} \\ U_\mathrm{F} \\ Y_\mathrm{F} \end{bmatrix} g = \begin{bmatrix} u_\mathrm{ini} \\ y_\mathrm{ini} \\ u \\ y \end{bmatrix}$$

$$u \in \mathcal{U}, y \in \mathcal{Y}$$

Applications:

Power System (Huang, 2022)



Connected Vehicle (Wang, 2023)

Multi-agent System (Fawcett, 2021)

Problem Statement

Adaptations for non-linear systems:

1) More data to increase accuracy
2) Extra slack variables to ensure feasibility
3) Different Regularization terms to increase the control performance

$$\min_{g,\sigma_y,u\in\mathcal{U},y\in\mathcal{Y}} \sum_{k=t}^{t+N-1} \left(\|y(k)\|_Q^2 + \|u(k)\|_R^2\right) + \mathrm{reg}$$

$$\text{subject to} \quad \begin{bmatrix} \mathbf{U}_\mathrm{P} \\ \mathbf{Y}_\mathrm{P} \\ \mathbf{U}_\mathrm{F} \\ \mathbf{Y}_\mathrm{F} \end{bmatrix} g = \begin{bmatrix} u_\mathrm{ini} \\ y_\mathrm{ini} + \sigma_y \\ u \\ y \end{bmatrix}$$

Existing Methods:

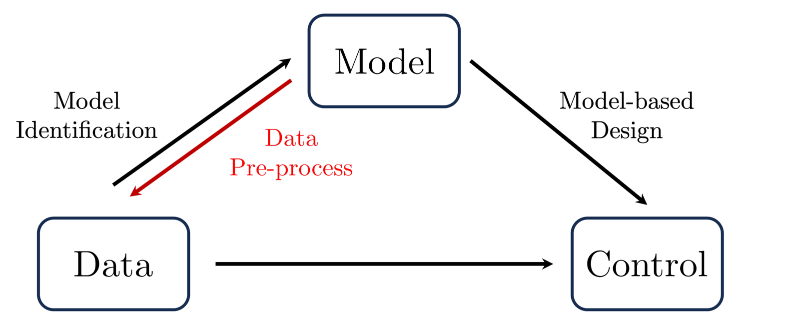
- **Regularization terms**: l_1-norm, l_2-norm, projection norm
- **Dimension reduction**: singular value decomposition (SVD), kernel representation

In this paper, we are interested in:

- How to combine different variants in a same framework?
- What are their relationships?
- How to further improve the Data-EnablEd predictive control?

Convex Approximation for a Bi-level Formulation

Key Insight:
Model and data are coupled: we can pre-process the data based on the corresponding model



Bi-level formulation (Dörfler, 2022):
Inner: data pre-processing, cost function and constraints developed from system ID
Outer: predictive control, extra slack variables for handling model mismatch

minimize: control cost (u, y)
subject to: (u, y) consistent with $(\bar{u}_\mathrm{d}, \bar{y}_\mathrm{d})$
where: $(\bar{u}_\mathrm{d}, \bar{y}_\mathrm{d})$ is the optimal solution of
minimize: Pre-processing Cost $(\bar{u}_\mathrm{d}, \bar{y}_\mathrm{d}, u_\mathrm{d}, y_\mathrm{d})$
subject to: Constraints from system identification:
Row Space, Rank Number, Hankel Structure

$$\underset{g,\sigma_y,u\in\mathcal{U},y\in\mathcal{Y}}{\text{minimize}} \sum_{k=t}^{t+N-1} \left(\|y(k) - y_r(k)\|_Q^2 + \|u(k)\|_R^2\right) + \lambda_y\|\sigma_y\|_2^2$$

$$\begin{aligned}\text{subject to}\quad & \tilde{H}^* g = \mathrm{col}(u_\mathrm{ini}, y_\mathrm{ini} + \sigma_y, u, y), \\ \text{where}\quad & \tilde{H}^* \in \arg\min_{\tilde{H}} \ J(\tilde{H}, H), \\ \text{subject to}\quad & \tilde{Y}_\mathrm{F} = Y_\mathrm{F}/\mathrm{col}(\tilde{U}_\mathrm{P}, \tilde{Y}_\mathrm{P}, \tilde{U}_\mathrm{F}) \quad \text{(Row Space)}, \\ & \mathrm{rank}(\tilde{H}) = mL + n \quad \text{(Rank Number)}, \\ & \tilde{H} \in \mathcal{H} \quad \text{(Hankel Structure)}.\end{aligned}$$

Method 1: DeePC-Hybrid (Dörfler, 2022):
DeePC-Hybrid relaxes the rank constraint and the row space constraint while keeping the Hankel structure

$$\min_{g,\sigma_y,u\in\mathcal{U},y\in\mathcal{Y}} \|u\|_R^2 + \|y\|_Q^2 + \lambda_1\|g\|_1 + \lambda_2\|(I - \Pi_1)g\|_2^2 + \lambda_y\|\sigma_y\|_2^2$$

$$\text{subject to} \quad \begin{bmatrix} U_\mathrm{P} \\ Y_\mathrm{P} \\ U_\mathrm{F} \\ Y_\mathrm{F} \end{bmatrix} g = \begin{bmatrix} u_\mathrm{ini} \\ y_\mathrm{ini} + \sigma_y \\ u \\ y \end{bmatrix}$$

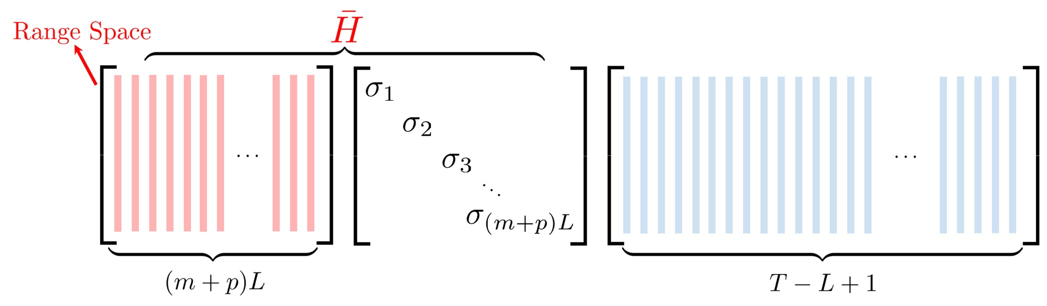
Rank:
$\mathrm{rank}(\tilde{H}) = mL + n$
Row Space:
$\tilde{Y}_F = Y_F/\mathrm{col}(\tilde{U}_P, \tilde{Y}_P, \tilde{U}_F)$

Method 2: DeePC-SVD (Zhang, 2023):
DeePC-SVD decreases the column dimension of pre-collected trajectory library by utilizing singular value decomposition (SVD) and drops Hankel structure

$$\min_{\bar{g},\sigma_y,u\in\mathcal{U},y\in\mathcal{Y}} \|u\|_R^2 + \|y\|_Q^2 + \lambda_1\|\bar{g}\|_1 + \lambda_2\|(I - \bar{\Pi}_1)\bar{g}\|_2^2 + \lambda_y\|\sigma_y\|_2^2$$

$$\text{subject to} \quad \begin{bmatrix} \bar{U}_\mathrm{P} \\ \bar{Y}_\mathrm{P} \\ \bar{U}_\mathrm{F} \\ \bar{Y}_\mathrm{F} \end{bmatrix} \bar{g} = \begin{bmatrix} u_\mathrm{ini} \\ y_\mathrm{ini} + \sigma_y \\ u \\ y \end{bmatrix}$$

$$H = \mathrm{col}(U_P, Y_P, U_F, Y_F) = W\Sigma V^\top$$
$$\bar{H} = W\Sigma = \mathrm{col}(\bar{U}_P, \bar{Y}_P, \bar{U}_F, \bar{Y}_F)$$



Method 3: Data-Driven-SPC:
Data-Driven-SPC relaxes the rank constraint, drops the Hankel structure but directly handles the row space constraint

- **Inner problem**

$$\min_{\tilde{H}} \ \|\mathrm{col}(\tilde{U}_\mathrm{P}, \tilde{Y}_\mathrm{P}, \tilde{U}_\mathrm{F}) - \mathrm{col}(U_\mathrm{P}, Y_\mathrm{P}, U_\mathrm{F})\|$$
$$\text{subject to} \quad \tilde{Y}_\mathrm{F} = Y_\mathrm{F}/\mathrm{col}(\tilde{U}_\mathrm{P}, \tilde{Y}_\mathrm{P}, \tilde{U}_\mathrm{F}),$$

- **Outer problem**

$$\min_{\sigma_y,g,u\in\mathcal{U},y\in\mathcal{Y}} \|u\|_R^2 + \|y\|_Q^2 + \lambda_1\|g\|_1 + \lambda_y\|\sigma_y\|_2^2$$

$$\text{subject to} \quad \begin{bmatrix} U_\mathrm{P} \\ Y_\mathrm{P} \\ U_\mathrm{F} \\ M \end{bmatrix} g = \begin{bmatrix} u_\mathrm{ini} \\ y_\mathrm{ini} + \sigma_y \\ u \\ y \end{bmatrix}.$$

Method 4: DeePC-SVD-Iter:
DeePC-SVD-Iter relaxes row space constraint but directly handles rank constraint and approximates the Hankel structure

- **Inner problem**

$$\min_{\tilde{H}} \ \|\tilde{H} - H\|_2$$
$$\begin{aligned}\text{subject to}\quad & \mathrm{rank}(\tilde{H}) = mL + n \\ & \tilde{H} \in \mathcal{H}\end{aligned}$$

- **Outer problem**

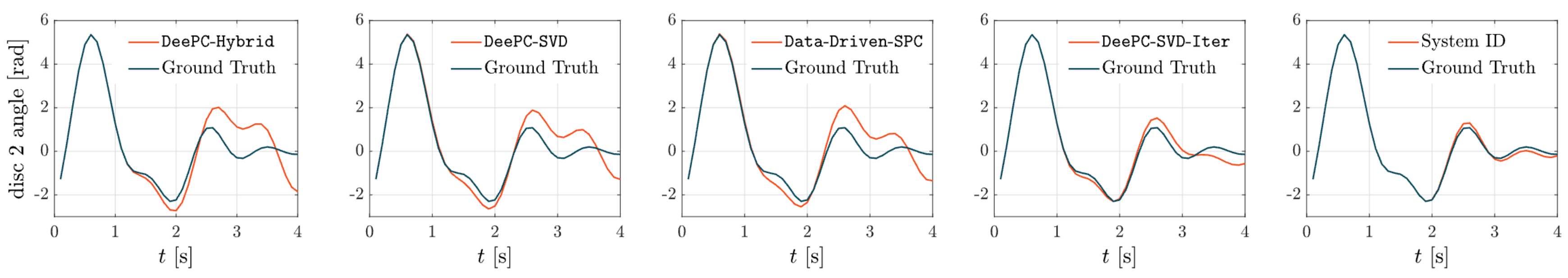
$$\min_{\hat{g},\sigma_y,u\in\mathcal{U},y\in\mathcal{Y}} \|u\|_R^2 + \|y\|_Q^2 + \lambda_2\|(I - \hat{\Pi}_1)\hat{g}\|_2^2 + \lambda_y\|\sigma_y\|_2^2$$

$$\text{subject to} \quad \begin{bmatrix} \hat{U}_\mathrm{P} \\ \hat{Y}_\mathrm{P} \\ \hat{U}_\mathrm{F} \\ \hat{Y}_\mathrm{F} \end{bmatrix} \hat{g} = \begin{bmatrix} u_\mathrm{ini} \\ y_\mathrm{ini} + \sigma_y \\ u \\ y \end{bmatrix}$$

Numerical Simulations

LTI system with Gaussian measurement noises:

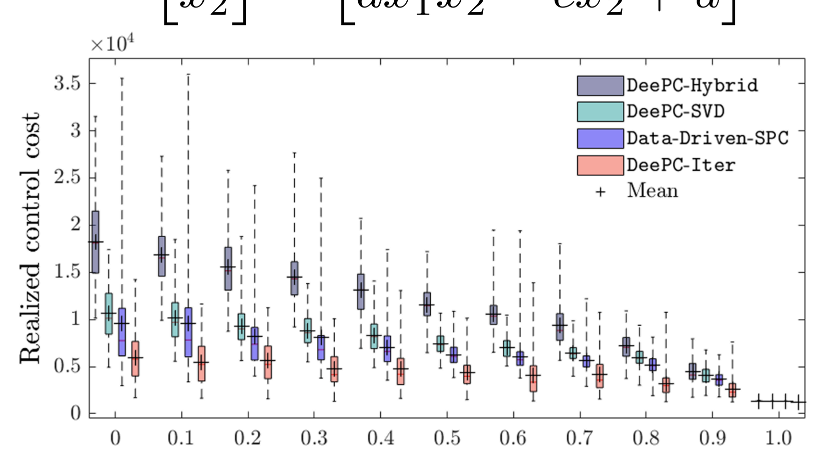
$$\begin{cases} x(k+1) = Ax(k) + Bu(k), \\ \quad y(k) = Cx(k) + Du(k) + \omega(k). \end{cases} \qquad \omega \sim \mathcal{N}(0, 0.01I)$$



(a) Hybrid (b) SVD (c) DDSPC (d) SVD-Iter (e) System ID

Nonlinear Lotka-Volterra dynamics:

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} ax_1 - bx_1x_2 \\ dx_1x_2 - cx_2 + u \end{bmatrix}.$$



Comparison of realized control cost for different approaches

Key observations:

- The cost becomes smaller as the data-driven representation becomes more structured
- The direct methods outperform the indirect method as the increasing of nonlinearity
- Among all DeePC variants, *DeePC-SVD-Iter* performs the best
- Regularizations are soft constraints and flexible while data pre-process with system knowledge gives hard requirements which makes the prediction more reliable

1. Shang, Xu, and Yang Zheng. "Convex approximations for a bi-level formulation of data-enabled predictive control." *6th Annual Learning for Dynamics & Control Conference*, 2024.
2. Dörfler, Florian, Jeremy Coulson, and Ivan Markovsky. "Bridging direct and indirect data-driven control formulations via regularizations and relaxations." *IEEE Transactions on Automatic Control*, 2022.